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The finite deformation of a reinforced packer, membrane theory

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Abstract. Using a membrane theory, an analytical solution is reached for the deformation of a packer membrane reinforced with inextensible cords. This model gives the deformation of the membrane and the levels of strain and stress in the membrane and the cords. The model enables one to study the influence of the various design parameters on the behaviour and integrity of the packer. Based on the outputs of the model, packer design suggestions are made which are expected to be of practical significance.

Notation

$2h_0$ initial membrane thickness (L); W strain energy ($ML^{-1}T^{-2}$); λ_1 meridian extension; λ_2 radial extension; Λ_2 maximum λ_2, i.e. packer expansion ratio; $2l_0$ initial membrane length (L); P relative inflation pressure ($ML^{-1}T^{-2}$); τ tension in cords (MLT^{-2}); Δ initial spacing between cords (L)

1. Introduction

To carry out pressure testing or hydraulic fracturing tests in an oil well, composite inflatable packers are often used to seal off and apply specified pressure to the wall of either uncased or cased wellbores. To undergo the requisite expansion (3:1 expansion), the packers need to be designed to be failsafe and reusable at high pressures. This paper is concerned with a membrane analysis of composite packers under actual downhole conditions and serves as an example of how the large deformation theory of reinforced composites can be applied.

To attempt a relatively simple analysis of the deformation of the packer as it is deformed downhole, we discuss below the consequences of applying a membrane theory of a fibre reinforced rubber packer. The basic theory for reinforced membranes has been outlined by Kydoniefs [1]. This theory enables one to calculate the expanded shape of the membrane as a function of the internal pressure given the expression for the strain energy function of the elastomer (this is to be obtained from experiment) and the shape and distribution of the cords. The theory also enables the tension in the cords and in the elastometer material to be calculated.

2. General equations

Axisymmetric deformations are considered of an initially cylindrical membrane of uniform thickness $2h_0$ composed of an elastic homogeneous isotropic and incompressible material possessing a strain energy function $W = W(I_1, I_2)$ and reinforced by two families of perfectly flexible and inextensible cords.

The cords of the two families are assumed to form constant angles $\pm\alpha$ with the generators of the undeformed membrane. It is also assumed that the lengths of the intercepts on any one cord of the one family by two adjacent cords of the other family are independent of the position on the surface and that these lengths are small compared with the radii of curvature at any point of the deformed or undeformed membrane. Moreover, it is assumed that no two cords of the same family are brought into contact as a result of the deformation and that intersecting cords of the two families do not move relative to each other at their point of intersection.

As any cylindrical membrane can be deformed into a circular cylinder without extension on its surface, it can be assumed, without loss of generality, that the undeformed membrane is a circular cylinder.

We refer the deformation to cylindrical polar coordinates and denote by (ρ, θ, η), $\rho =$ constant, the coordinates in the undeformed configuration of a point which has coordinates (r, θ, z) in the deformed state. Because the deformation is axially symmetric

$$r = r(\eta), \qquad z = z(\eta). \tag{1}$$

The elements of length in the undeformed and deformed configurations will be denoted by dS and ds, respectively. The angle of dS with the generator of the undeformed membrane will be denoted by a and the element of the deformed meridian by dξ. It can then be shown [1, 2] that

$$\left(\frac{ds}{dS}\right)^2 = \lambda_1^2(\cos(a))^2 + \lambda_2^2(\sin(a))^2 \tag{2}$$

where the notation

$$\lambda_1 = d\xi/d\eta, \qquad \lambda_2 = r/\rho \tag{3}$$

has been used.

From the symmetry of the system and the incompressibility of the material it follows that λ_1, λ_2 and $\lambda_3 = (\lambda_1\lambda_2)^{-1}$ are the principal extension ratios in terms of which the strain invariants I are given by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{4}$$

$$I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} \tag{5}$$

Since the cords are inextensible, for $a = \alpha$ we obtain

$$\lambda_1^2(\cos(\alpha))^2 + \lambda_2^2(\sin(\alpha))^2 = 1 \tag{6}$$

which gives λ_1 in terms of λ_2 and α.

Let $\pm\beta(\eta)$ be the angles of the two families of cords with the meridian of the deformed section. Then, if dS, ds are on the same cord,

$$r\sin(\alpha)\, dS = \rho \sin(\beta)\, ds \tag{7}$$

and

$$\cos(\beta)\, ds = d\xi = \lambda_1 \cos(\alpha)\, dS \tag{8}$$

From these relations we obtain

$$\sin(\beta) = \lambda_2 \sin(\alpha) \tag{9}$$

and

$$\cos(\beta) = \lambda_1 \cos(\alpha) \tag{10}$$

Let Δ and δ denote the distance between adjacent cords in the undeformed and deformed membrane, respectively. From

$$\Delta = \rho \cos(\alpha)\, d\theta\ , \quad \delta = r \cos(\beta)\, d\theta$$

and the above relations for sin and cos β we get

$$\delta = \lambda_1 \lambda_2 \Delta \tag{11}$$

A further assumption is that the meridian C of the deformed membrane does not intersect the z-axis and it has no finite element parallel or perpendicular to the z-axis. Moreover, and without loss of generality, it is assumed that the tangent to C is nowhere parallel or perpendicular to the z-axis, except perhaps at an end point of C; otherwise C may be divided into segments for which this condition does hold. This condition will be relaxed somewhat when we consider the situation when the expander meets the casing. With this assumption the equations of equilibrium

$$\frac{d}{d\xi}(rT_1) = T_2 \frac{dr}{d\xi} \tag{12}$$

and

$$\kappa_1 T_1 + \kappa_2 T_2 = P \tag{13}$$

which are valid for any axisymmetric deformation of a membrane, can be written in the form

$$\frac{d}{dr}(rT_1) = T_2 \tag{14}$$

and

$$\kappa_1 T_1 + \kappa_2 T_2 = P \tag{15}$$

where T_1, T_2 are the stress resultants, per unit length of the deformed membrane, in the directions of the meridian and the circle of latitude, respectively, κ_α the principal curvatures, κ_1 being the curvature of the meridian, and P is the internal pressure. If $\omega < \pi/2$ denotes the angle formed by the tangent to the meridian of the deformed membrane and the axis of symmetry then

$$\kappa_1 = \frac{d}{dr}\cos(\omega) \tag{16}$$

$$\kappa_2 = \frac{\cos(\omega)}{r} \tag{17}$$

By using (3), (16) and (17) the equations (14) and (15) are reduced to

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_2}(\lambda_2 T_1) = T_2 , \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda_2}(\lambda_2 T_1 \cos(\omega)) = P\rho\lambda_2 \tag{18}$$

The stress resultants T_α can be resolved into two parts:

$$T_\alpha = T'_\alpha + T''_\alpha \tag{19}$$

where T'_α is due to the deformation of the material and can be expressed in terms of the strain energy function W by

$$T'_1 = 4h_0\lambda_3(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2 \frac{\partial W}{\partial I_2}\right) = \frac{2h_0}{\lambda_2}\frac{\partial W}{\partial \lambda_1} \tag{20}$$

and

$$T'_2 = 4h_0\lambda_3(\lambda_2^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2 \frac{\partial W}{\partial I_2}\right) = \frac{2h_0}{\lambda_1}\frac{\partial W}{\partial \lambda_2} \tag{21}$$

and T''_α is due to the tension in the cords. These expressions can be written in the form

$$T''_1 = \frac{2\lambda_1 \cos^2\alpha}{\lambda_2 \Delta}\tau , \quad T''_2 = \frac{2\lambda_2 \sin^2\alpha}{\lambda_1 \Delta}\tau \tag{22}$$

By using (19) to (22) the first equation of equilibrium (18) can be solved [3] giving

$$\tau = \frac{\Delta h_0(W - \lambda_1(\partial W/\partial \lambda_1)) + A}{1 - \lambda_2^2 \sin^2\alpha} \tag{23}$$

where A is an integration constant. From the definition of the invariants (4) and (5) together with λ_1 given in terms of λ_2 from (6) and the expressions (22) and (23) we get T''_α as functions of λ_2. This together with the expressions (20) and (21) give the forces T_α in terms of λ_2 and the integration constant A. It remains to integrate the second of equations (18) to obtain ω in terms of λ_2 for a given set of boundary conditions.

2.1. Applications

As indicated above we still need to integrate the second of (18) to proceed with the solution to a given problem. For a membrane loaded under constant pressure P_0 this gives

$$\lambda_2 T_1 \cos(\omega) = \frac{P\rho}{2}(\lambda_2^2 - C) \tag{24}$$

where C is an arbitrary constant. For the free end condition on the packer $\lambda_2 = 1$ and the corresponding force in the axial direction should be zero. Hence

$$T_1 \cos(\omega) = 0 \tag{25}$$

ıen $\lambda_2 = 1$. Thus the constant

$$C = 1 . \tag{26}$$

Since in addition $\omega = 0$ when $\lambda_2 = \Lambda_2$, the maximum expansion ratio of the membrane for a given pressure, then (24) can be written

$$\Lambda_2(T_1)_{\Lambda_2} = \frac{P\rho}{2}(\Lambda_2^2 - 1) \tag{27}$$

Recall from equations (20) and (22) that

$$\lambda_2 T_1 = 2h_0 \frac{\partial W}{\partial \lambda_1} + \frac{2\lambda_1 \cos^2\alpha}{\Delta}\tau \tag{28}$$

This gives the constant A using (23) and (27) then (24) gives $\cos(\omega)$ and the shape and other properties of the membrane follow by performing the integrals outlined below and solving, if necessary, the appropriate non-linear equation. For the case of a membrane expanding freely this amounts to solving the equation for the specified initial length of the membrane.

The strain energy function

To proceed further we need an expression for the energy density of the elastomer. This has been deduced from experimental tests. A representative, in SI units, is

$$W = 0.24\,10^6(I_1 - 3) + 0.0084\,10^6(I_1 - 3)^2 \tag{29}$$

Using this expression we can deduce the force functions in the elastomer from the equations (20) and (21).

2.2. *The shape of the expanded membrane*

Once the angle ω has been obtained from (24) the shape of the membrane can be determined provided the maximum expansion Λ_2 has been found. For the cases of a membrane expanding under a specified pressure the condition that determines Λ_2 is that the initial length of the membrane is specified as $2l_0$ say. Then

$$\frac{l_0}{\rho} = -\int_{\Lambda_2}^{1} \frac{d\lambda_2}{\lambda_1 \sin(\omega)} . \tag{30}$$

To effect a solution of this equation, it is helpful to have an estimate of the range of values in which a solution may lie. For the case $\alpha = 0$, a lower bound for the pressure P to provide a maximum expansion ratio Λ_2 can be deduced as

$$P \geqslant \frac{4h_0}{\rho}\left(\frac{\partial W}{\partial(\lambda_2^2)}\right)_{\lambda_2^2 = \Lambda_2^2} . \tag{31}$$

The shape of the deformed membrane is determined by plotting λ_2, the expansion ratio r/ρ, versus z the axial coordinate in the deformed coordinate system where

$$\frac{z}{\rho} = \mp \int_{\Lambda_2}^{\lambda_2} \cot(\omega)\, d\lambda_2 \tag{32}$$

and the plus or minus sign is used according to whether λ_2 is an increasing or decreasing function of ξ and $\xi = 0$ for $z = 0$. The length ξ of the deformed (semi) meridian is given by

$$\frac{\xi}{\rho} = \mp \int_{\Lambda_2}^{1} \frac{d\lambda_2}{\sin(\omega)} . \tag{33}$$

As stated above for the free expansion of the packer one must solve equation (30) for the maximum expansion ratio Λ_2. However, for the situation, when the packer expands against a rigid casing the procedure is different [4]. For this case Λ_2 is specified and the half length of the central section which is in contact with the wall is given by

$$z_0 = \left(l_0 + \rho \int_{\Lambda_2}^{1} \frac{d\lambda_2}{\lambda_1 \sin(\omega)} \right) \lambda_1 . \tag{34}$$

In the above equation the factor λ_1 outside the integral sign is evaluated at the maximum expansion ratio Λ_2. The shape of the rest of the membrane is determined as above except that now z given by equation (32) is replaced by $z + z_0$.

3. Numerical results and engineering considerations

To give some perspective on the kind of results predicted by the membrane theory, three cases of inflation have been considered for four values of cords angle α. The influence of α is of particular interest from a design stand point. For each of the four values of $\alpha \in \{0°, 10°, 12°, 15°\}$ the packer has been inflated up to the point where the membrane touches a borewall corresponding to an expansion ratio of $\lambda_2 = 3$. The computation has then been extended to touch-wall pressure plus 0.1 MPa and touch-wall pressure plus 10.0 MPa for each value of α as simulation of the operational working condition of the packer.

The numerical values used in the following examples were for a 10 cm ($\rho = 0.05$) diameter 1 m ($l_0 = 0.5$) long packer with a 1 cm ($h_0 = 0.005$) thick membrane of material represented by the strain energy function (29) reinforced with cords spaced every 5 mm ($\Delta = 0.005$). Δ appears only in the final computation of τ and δ.

Note that the angle β of the cords in the deformed configuration can be determined from the relation (9) i.e.

$$\sin(\beta) = \lambda_2 \sin(\alpha) . \tag{35}$$

once the expansion ratio λ_2 has been determined as a function of position on the membrane.

Inflation pressure and cords tension

Increasing values of α require increasing values of the packer inflation pressure P to reach the same expansion ratio $\lambda_2 = 3$. Figure 1a shows the shape of the unconstrained deformed membrane at the four values of α. The respective inflation pressures (0.143, 0.190, 0.237 and 0.639 MPa) increase by 33%, 66% and 347% with respect to the inflation pressure for $\alpha = 0°$. This effect, combined with the direct effect of α (τ is roughly proportional to $1/\cos\beta$), is detrimental to the performance of the packer since it results in an increased level of tension

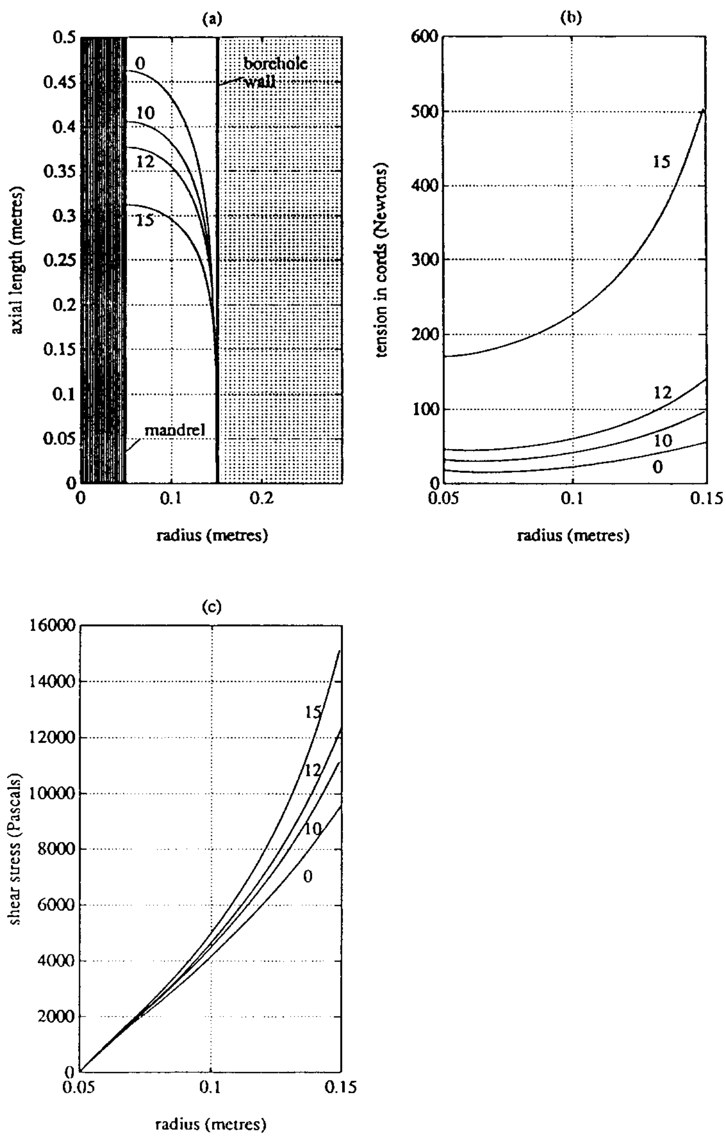

Fig. 1. P = 'touch-wall' pressure.

in the cords. Figure 1b shows the maximum tension in the cords τ increasing by 82%, 154% and 809% again with respect to the cords tension for $\alpha = 0°$. Note that the highest level of cords tension is reached at the point where the membrane meets the wall.

As the inflation pressure is increased beyond the touch-wall value, the membrane is constrained by the borewall; the maximum shear stress remains constant (Figs. 1c, 2c and 3c) since it is controlled by the expansion ratio, itself limited by the borewall; the tension in the cords increases and, although the relative influence of α decreases, its effect remains substantial even at pressure as high as touch-wall pressure plus 10 MPa (+20%, +34% and +74%; Fig. 3b). Also worth mentioning is the effect of α on the half-length of contact between the membrane and the borewall which is reduced from 0.359 m to 0.295 m (−18%) to 0.262 m (−27%) and to 0.175 m (−51%) for the four respective values of α at the high inflation pressure (Fig. 3a).

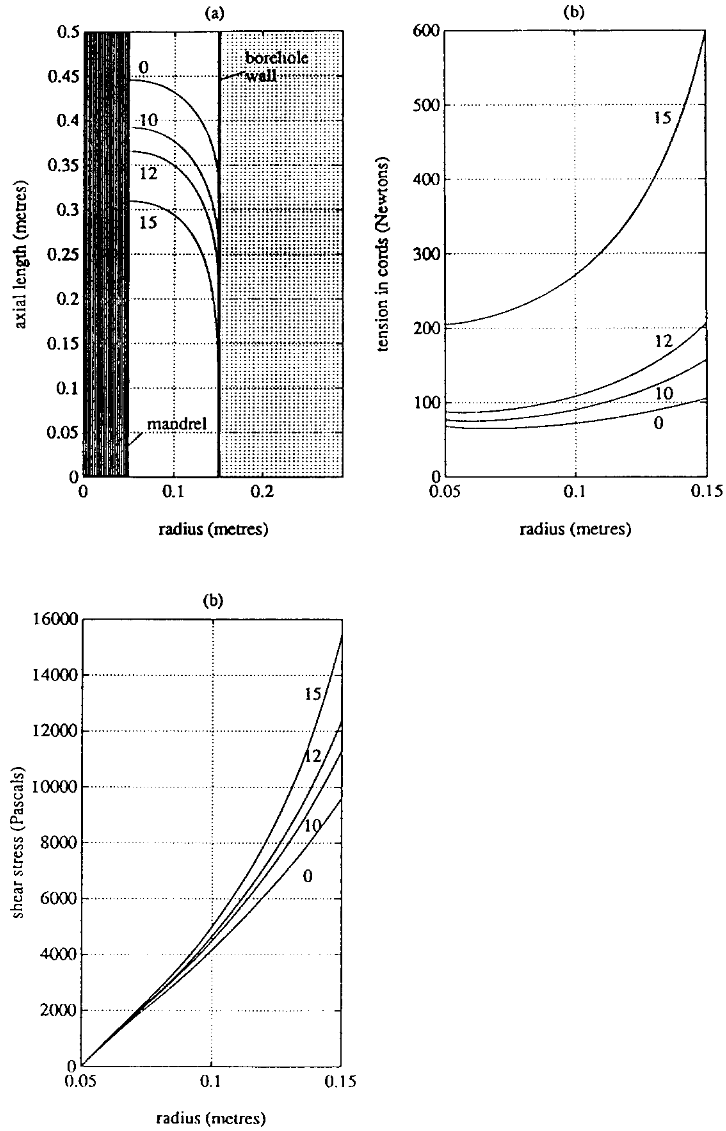

Fig. 2. P = 'touch-wall' pressure + 0.1 MPa.

The difference between the inflation pressure and the touch-wall pressure is the pressure with which the membrane is pressed against the borewall (neglecting the borewall deformation). This is the pressure which energises the sealing action of the packer. A low touch-wall pressure will therefore result in a lower final inflation pressure for a same sealing action.

Elastomer shear

Increasing values of α result in increasing values of shear strain γ in the elastomer although this effect is not pronounced for low values of α. Figure 4 shows γ plotted versus $\lambda_2 \in [1, 3]$ for five values of $\alpha \in \{0°, 10°, 12°, 15°, 20°\}$. Going from $\alpha = 0$ to $\alpha = 10°$ increases the shear strain by 10% for an expansion ratio of 3. As seen in the figure, this effect increases dramatically for higher but less relevant values of α. Note the high level of shear ($\gamma = 0.78$ ($\pi/4$) corresponds to an angle of 90° being folded to 45°) imposed on the elastomer at an

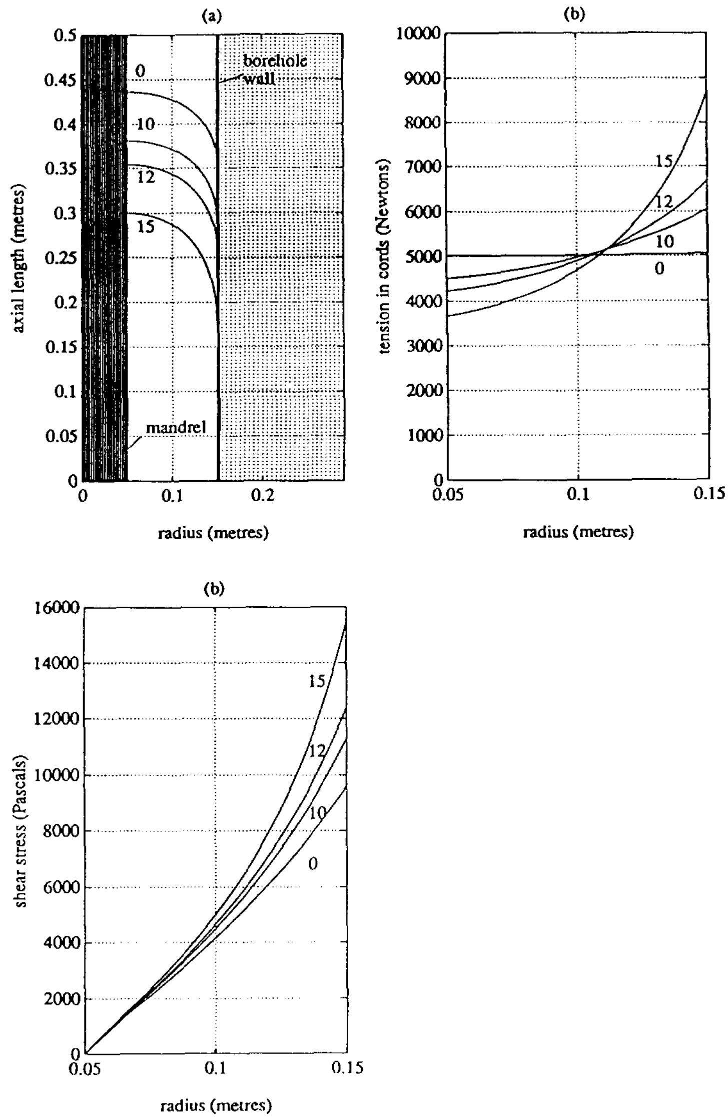

Fig. 3. P = 'touch-wall' pressure + 10 MPa.

expansion ratio of $\lambda_2 = 3$ even for $\alpha = 0$. The shear strain is the angle (in radian) by which a $\pi/2$ angle is deformed i.e.:

$$\gamma_{xy} = \frac{\pi}{2} - 2\arctan\left(\frac{\lambda_1}{\lambda_2}\right) \tag{36}$$

which combined with (6) yields the following expression of γ as a function of α and λ_2.

$$\gamma_{xy} = \frac{\pi}{2} - 2\arctan\left(\frac{\sqrt{1-\lambda_2^2\sin^2\alpha}}{\lambda_2\cos\alpha}\right) \tag{37}$$

The resulting shear stress in the elastomer are presented in Figs. 1c, 2c and 3c.

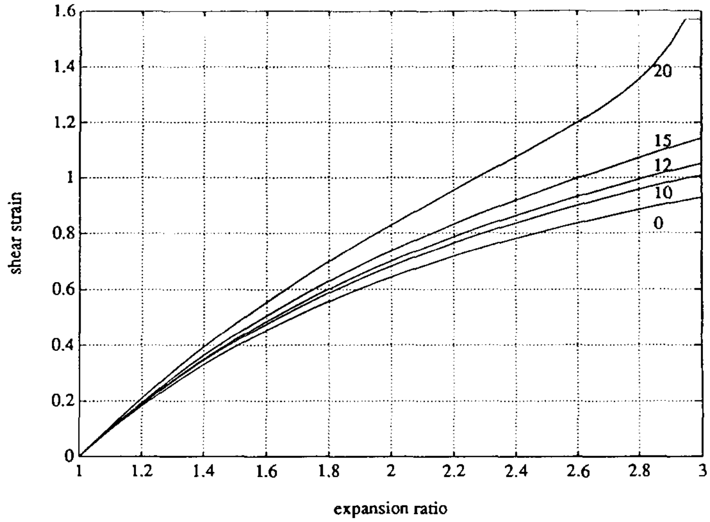

Fig. 4. Shear strain vs. expansion ratio.

Note that the value of α determines the maximum expansion ratio geometrically possible (6):

$$\lambda_{2_{max}} = \frac{1}{\sin\alpha} \tag{38}$$

$$\lambda_{2_{max}} = 3 \Leftrightarrow \alpha = 19.5° \tag{39}$$

Multi layered cords

It should be noted that the two layers of cords crossing each other ($\alpha \neq 0$) is a design feature which is not required for the integrity of the packer. The cord reinforcement is required to support the high meridian tension which takes place in the membrane sections located in the annulus between the borewall and the packer mandrel. This meridian reinforcement restrains the membrane from expanding axially; no azimuthal (hoop) reinforcement is needed since the radial expansion is constrained by the borewall. Therefore, a value of $\alpha = 0$ is acceptable and even desirable in view of the previous considerations on the influence of α on the inflation pressure P, the cords tension τ and the elastomer shear strain γ.

From equation (11) and (6) the spacing between cords is

$$\delta = \Delta\lambda_1\lambda_2 = \Delta\frac{\lambda_2\sqrt{1-\lambda_2^2\sin^2\alpha}}{\cos\alpha} \tag{40}$$

The derivative of δ by λ_2

$$\frac{d\delta}{d\lambda_2} = \frac{\Delta}{\cos\alpha}\frac{1-2\lambda_2^2\sin^2\alpha}{\sqrt{1-\lambda_2^2\sin^2\alpha}} \tag{41}$$

shows the presence of a maximum

Table 1. Maximum spacing between cords

α	λ_2^*	$(\lambda_1\lambda_2)_{max}$	$(\lambda_1\lambda_2)_{\lambda_2=3}$
0°	$d\delta/d\lambda_2 > 0\ \forall\lambda_2$	λ_2	3.00
10°	4.07	2.92	2.60
12°	3.4	2.46	2.40
15°	2.7	2.00	1.96

$$\lambda_2 = \frac{1}{\sqrt{2}\sin\alpha}\,. \tag{42}$$

The influence of α on the maximum spacing between cords δ_{max} is shown in Table 1. The second column shows the value λ_2^* at which δ_{max} occurs.

Two layers of closely spaced cords are often used in order to limit the extrusion of the elastomer between cords. Unfortunately, this method tends to defeat its own purpose: closely spaced cords leave little elastomer material in-between cords resulting in destructive level of shear in the surrounding elastomer. This leads to the debonding of the cords which then are no more maintained equally spaced and tend to bundle leaving large gaps of unsupported elastomer subject to extrusion under high inflation pressure. This has been shown experimentally to be the case by Newburn [5].

Design suggestions

In view of these considerations the following design approach is proposed. The membrane is to be composed of a single layer of cords laid along the meridian of the packer ($\alpha = 0$). The spacing between the cords is to be such as to leave space for a hole or a notch between each two cords as shown in Fig. 5 in order to limit the level of strain due to λ_2 in the elastomer in-between the cords. This feature is to avoid debonding of the cords and thereby maintain the cords equally spaced.

Experimental evidence [6] indicates that in-homogeneities in the elastomer can result in non-uniform stretching of the elastomer leading to nonequal spacing of the cords and

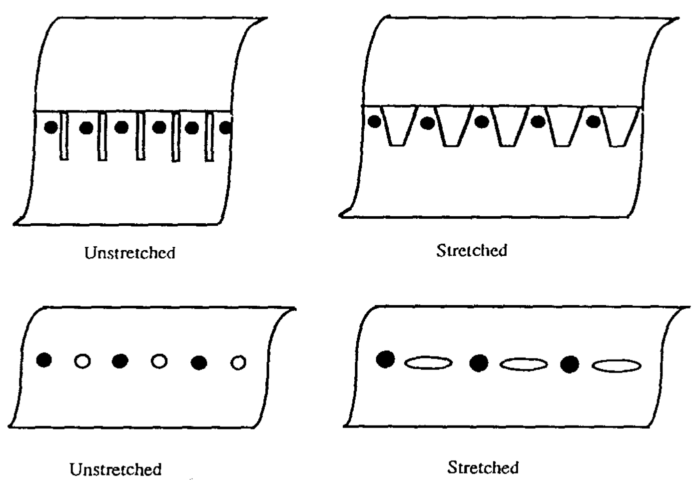

Fig. 5.

extrusion of the elastomer. Although it is expected that the hole or notch arrangement will by itself be sufficient to keep the cords equally spaced, as a further remedy to the problem it might be useful to constrain the maximum spacing between cords by a series of flexible wires placed radially in the hole or notch between cords. As an example, a maximum packer expansion ratio of 3 would lead to a length of wire limiting the local expansion ratio to, say, 3.3.

Borewall shear

The two ends of the packer are free to move axially; one end is built to slide along the packer mandrel; the other end is fixed to the mandrel which can move axially with little restraint from the compliant string holding the packer tool in the hole. As a consequence, the mandrel offers no axial support to the membrane and the axial force resulting from the pressure across the membrane over the annulus section is entirely supported by the friction between the membrane and the borewall. When a high differential pressure is imposed across the packer, the axial force applied on the packer results in high level of shear stress at the interface between the membrane and the borewall which could lead, especially with openhole in weak formations, to slippage, high deformation and failure of the packer.

4. Conclusion

Using a membrane theory, an analytical solution has been reached for the deformation of a packer membrane reinforced with inextensible cords. This model gives the deformation of the membrane and the levels of strain and stress in the membrane and the cords. Although, as it is, the model shows the influence of the various design parameters on the behaviour and integrity of the packer, the model should now be checked against experimental data to ascertain the limit of its validity.

The main issue for the integrity of the membrane seems to be the extrusion of the elastomer in-between un-equally spaced cords. The traditional design of two layers of cords crossing each other does not guarantee equally spaced cords in the inflated state because of debonding of the cords. It is suggested that a single layer of cords laid along the meridian of the packer and inbedded in the elastomer with some added compliance in-between the cords (hole or notch) is more likely to offer a solution.

Acknowledgement

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